

# Hawking Radiation inside Black Holes in Quantum Gravity

Ichiro Oda <sup>1</sup>

Department of Physics, Faculty of Science, University of the Ryukyus,  
Nishihara, Okinawa 903-0213, Japan.

## Abstract

We study black hole radiation inside black holes within the framework of quantum gravity. First, we review on our previous work of a canonical quantization for a spherically symmetric geometry where one of the spatial coordinates is treated as the time variable, since we think of the interior region of a black hole. Based on this formalism, under physically plausible assumptions, we solve the Wheeler-De Witt equation inside the black hole, and show that the mass-loss rate of an evaporating black hole due to thermal radiation is equivalent to the result obtained by Hawking in his semi-classical approach. A remarkable point is that our assumptions make the momentum constraint coincide with the Hamiltonian constraint up to an irrelevant overall factor. Furthermore, for comparison, we solve the Wheeler-De Witt equation outside the black hole as well, and see that the mass-loss rate of an evaporating black hole has the same expression. The present analysis suggests that the black hole radiation comes from the black hole singularity. We also comment on the Birkhoff theorem in quantum gravity.

---

<sup>1</sup>E-mail address: ioda@phys.u-ryukyu.ac.jp

# 1 Introduction

There has been a recent revival of an interest in the interior of a black hole. For instance, there has been an active debate on whether the AdS/CFT correspondence could describe the physics of the interior of a black hole or not [1, 2, 3]. This problem is closely related to the information loss paradox in physics of black holes. Moreover, it has been more recently pointed out that black holes have a very large interior: For a stellar black hole, the volume inside the black hole is larger than that of our universe [4, 5]. This study is also relevant to the information loss paradox since there might be a lot of real estate available inside a black hole for stocking micro-states associated with a black hole entropy.<sup>2</sup>

In the 1990's, there were also active studies of the interior region of a black hole despite that the interior is physically of no relevance for external observers outside the horizon. In those days, the interest in the interior of a black hole has been triggered by development of the understanding of the internal geometry near the Cauchy horizon inside the Reissner-Nordstrom black hole, what is called, the mass inflation [7, 8, 9], and the phenomenon of the smearing of a black hole singularity in quantum gravity [10]. As one of motivations behind these studies, there was an expectation that since both the Cauchy horizon and a spacetime singularity exhibit highly pathological behavior in the classical theory of general relativity, and quantum effects would play a dominant role, studies of the physics inside the horizon of a black hole might give us some important clues for constructing a theory of quantum gravity.

Stimulated with the interest in the interior of a black hole in the 1990's, we have already formulated a canonical formalism of a system with a spherically symmetric black hole holding in the interior region bounded by the apparent horizon and the singularity [11], which is a natural generalization of the canonical formalism holding in the exterior region covering the spacetime between the apparent horizon and the spatial infinity [12]. In this region, the Killing vector  $\frac{\partial}{\partial t}$  becomes spacelike whereas it is timelike in the exterior region. Consequently, one has to foliate the interior of a black hole with a family of spacelike hypersurfaces, for instance,  $r = \text{const.}$

As one of applications of this canonical formalism, following Tomimatsu's idea [13], we have considered black hole radiation in quantum gravity where it was shown that the mass-loss rate due to the black hole radiation is equal to that evaluated by Hawking in the semiclassical approximation [14].<sup>3</sup> In this work, we have focused on the vicinity of the apparent horizon where one component  $\gamma$  in the metric tensor becomes zero so we had to adopt a regularization such that  $\gamma$  takes a small but finite value.

This regularization is clearly so unwelcoming that one should dispense with it. In this article, without considering the vicinity of the apparent horizon at the beginning, making more physically plausible assumptions, we will derive the Hawking radiation in the interior region within the framework of quantum gravity. This modification of the model setting makes it possible to calculate the expectation value of the mass-loss rate without any pathology.

---

<sup>2</sup>See Ref. [6] for a phenomenological description of a spherically symmetric black hole.

<sup>3</sup>See also related works [15, 16].

Moreover, we will also consider the exterior region of a black hole and do the same job in order to make a comparison of black hole radiation between the interior and the exterior of the horizon.

The analysis of both the interior and exterior regions of the apparent horizon of a dynamical black hole suggests that the thermal radiation of the black hole comes from the spacetime singularity. This fact might imply that the resolution of the information loss paradox needs the understanding of the physics of the spacetime singularity of a black hole. As a bonus, we will comment on the Birkhoff theorem [17] in quantum gravity.

This article is organized as follows: In the next section, after mentioning notation and conventions, we review on the canonical formalism of a system with a spherically symmetric black hole in the interior region bounded by the apparent horizon and the singularity [11]. In Section 3, we apply the canonical formalism reviewed in Section 2 for the calculation of the mass-loss rate due to black hole radiation. In Section 4, we also calculate the mass-loss rate in the exterior region bounded by the apparent horizon and the spatial infinity. The final section contains a conclusion.

## 2 Review of canonical formalism inside black hole

Before delving into details, let us explain our notation and conventions. We mainly follow notation and conventions by Misner et al.'s textbook [18], for instance, the flat Minkowski metric  $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$ , the Riemann curvature tensor  $R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha}$ , and the Ricci tensor  $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ . Throughout this article, we adopt the natural units  $c = \hbar = G = 1$ . In this units, all quantities become dimensionless.

Let us start with the review of a canonical formalism of a spherically symmetric system with a black hole, which has been presented in our previous work [11], but compared to this work we will change the notation slightly for the purpose of the comparison with the canonical formalism in the exterior geometry (For instance, we have exchanged the role between the lapse function  $\alpha$  and the shift function  $\beta$ ).

As a first step, one needs to select arbitrary spherically symmetric spacelike hypersurfaces to foliate the spacetime. The key point here is that the radial coordinate plays the role of time in the interior of the horizon in the spherically symmetric coordinate system. As a simple choice, it is convenient to take the  $x^1 = \text{const}$  hypersurfaces to slice the interior region. In the next section, we will choose the simplest case,  $x^1 = r$ .

The four-dimensional action which we consider in this paper takes the following form:

$$S = \int d^4x \sqrt{-{}^{(4)}g} \left[ \frac{1}{16\pi} {}^{(4)}R - \frac{1}{4\pi} {}^{(4)}g^{\mu\nu} (D_\mu \Phi)^\dagger D_\nu \Phi - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \right], \quad (1)$$

where  $\Phi$  is a complex scalar field and its covariant derivative is given by

$$D_\mu \Phi = \partial_\mu \Phi + ie A_\mu \Phi, \quad (2)$$

with  $e$  and  $A_\mu$  being the electric charge of  $\Phi$  and the  $U(1)$  gauge field, respectively. Moreover, as usual,  $F_{\mu\nu}$  is the field strength defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (3)$$

To clarify the four-dimensional meaning we put the suffix (4) in front of the metric tensor and the scalar curvature. As a final note, the Greek indices  $\mu, \nu, \dots$  take the four-dimensional values 0, 1, 2 and 3 whereas the Latin ones  $a, b, \dots$  do the two-dimensional values 0 and 1. Of course, it is straightforward to include the other matter fields as well as the cosmological constant in the action (1) even if we limit ourselves to the action for simplicity.

The most general spherically symmetric ansatz for the four-dimensional line element is of form

$$\begin{aligned} {}^{(4)}ds^2 &= {}^{(4)}g_{\mu\nu}dx^\mu dx^\nu, \\ &= g_{ab}dx^a dx^b + \phi^2(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \quad (4)$$

where the two-dimensional metric  $g_{ab}$  and the radial function  $\phi$  are the functions of only the two-dimensional coordinates  $x^a$ . The substitution of the ansatz (4) into the action (1) and then integration over the angular variables  $(\theta, \varphi)$  produces the two-dimensional effective action

$$\begin{aligned} S &= \frac{1}{2} \int d^2x \sqrt{-g} \left[ 1 + g^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2} R \phi^2 \right] \\ &- \int d^2x \sqrt{-g} \left[ \phi^2 g^{ab} (D_a \Phi)^\dagger D_b \Phi + \frac{1}{4} \phi^2 F_{ab} F^{ab} \right], \end{aligned} \quad (5)$$

where we have assumed that  $A_a$  and  $\Phi$  are also the functions of the two-dimensional coordinates  $x^a$  and set  $A_\theta = A_\varphi = 0$ .

Next let us rewrite the action (5) in the ADM form. As remarked before, we will regard the  $x^1$  spatial coordinate as time to cover the interior of a black hole by spacelike hypersurfaces. The appropriate ADM splitting of (1+1)-dimensional spacetime is given by

$$g_{ab} = \begin{pmatrix} \gamma & \beta \\ \beta & \frac{\beta^2}{\gamma} - \alpha^2 \end{pmatrix}. \quad (6)$$

The normal unit vector  $n^a$  which is orthogonal to the hypersurfaces  $x^1 = \text{const}$  reads

$$n^a = \left( \frac{\beta}{\alpha\gamma}, -\frac{1}{\alpha} \right). \quad (7)$$

The induced metric on the hypersurfaces, that is, the projection operator over  $x^1 = \text{const}$  hypersurfaces, is given by

$$h^{ab} = g^{ab} + n^a n^b. \quad (8)$$

It is easy to check that  $h^{ab}$  is indeed the projection operator by inserting (6) and (7) to (8).

The extrinsic curvature  $K_{ab}$ , its trace  $K$  and the scalar curvature  $R$  are given by [19]

$$\begin{aligned} K_{ab} &= K_{ba} = h_a^c \nabla_c n_b, \\ K &= g^{ab} K_{ab} = \nabla_a n^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} n^a), \\ R &= 2n^a \partial_a K + 2K^2 - 2\nabla_c (n^a \nabla_a n^c). \end{aligned} \quad (9)$$

Using Eqs. (6)-(9), a straightforward calculation reveals us

$$\begin{aligned} K &= -\frac{\gamma'}{2\alpha\gamma} + \frac{\dot{\beta}}{\alpha\gamma} - \frac{\beta}{2\alpha\gamma^2} \dot{\gamma}, \\ R &= 2n^a \partial_a K + 2K^2 - \frac{2}{\alpha\sqrt{\gamma}} \partial_0 \left( \frac{\dot{\alpha}}{\sqrt{\gamma}} \right), \end{aligned} \quad (10)$$

where  $\frac{\partial}{\partial x^0} = \partial_0$  and  $\frac{\partial}{\partial x^1} = \partial_1$  are also denoted as an overdot and a prime, respectively. With the help of these equations, one can cast the action (5) to the form

$$\begin{aligned} S &\equiv \int d^2x L \\ &= \int d^2x \left[ \frac{1}{2} \alpha \sqrt{\gamma} \left\{ 1 - (n^a \partial_a \phi)^2 + \frac{1}{\gamma} \dot{\phi}^2 - K n^a \partial_a (\phi^2) + \frac{\dot{\alpha}}{\alpha\gamma} \partial_0 (\phi^2) \right\} \right. \\ &\quad + \left. \alpha \sqrt{\gamma} \phi^2 \left\{ |n^a D_a \Phi|^2 - \frac{1}{\gamma} |D_0 \Phi|^2 \right\} + \frac{1}{2} \alpha \sqrt{\gamma} \phi^2 E^2 \right] \\ &\quad + \int d^2x \left[ \frac{1}{2} \partial_a (\alpha \sqrt{\gamma} K n^a \phi^2) - \frac{1}{2} \partial_0 \left( \frac{\dot{\alpha}}{\sqrt{\gamma}} \phi^2 \right) \right], \end{aligned} \quad (11)$$

where we have defined  $E$  as

$$E = \frac{1}{\sqrt{-g}} F_{01} = \frac{1}{\alpha\sqrt{\gamma}} (\dot{A}_1 - A'_0). \quad (12)$$

Now the differentiation of the action (11) with respect to the spatial derivative of the canonical variables  $\Phi(\Phi^\dagger)$ ,  $\phi$ ,  $\gamma$  and  $A_0$  leads to the corresponding canonical conjugate momenta  $p_\Phi(p_{\Phi^\dagger})$ ,  $p_\phi$ ,  $p_\gamma$  and  $p_A$

$$\begin{aligned} p_\Phi &= -\sqrt{\gamma} \phi^2 n^a (D_a \Phi)^\dagger, \quad p_\phi = \sqrt{\gamma} n^a \partial_a \phi + \sqrt{\gamma} K \phi, \\ p_\gamma &= \frac{1}{4\sqrt{\gamma}} n^a \partial_a (\phi^2), \quad p_A = -\phi^2 E. \end{aligned} \quad (13)$$

Then, the Hamiltonian, which is defined as

$$H = \int dx^0 \left( p_\Phi \Phi' + p_{\Phi^\dagger} \Phi'^\dagger + p_\phi \phi' + p_\gamma \gamma' + p_A A'_0 - L \right), \quad (14)$$

is expressed in terms of a linear combination of constraints as expected from diffeomorphism invariance

$$H = \int dx^0 (\alpha H_0 + \beta H_1 + A_1 H_2), \quad (15)$$

where  $\alpha, \beta$  and  $A_1$  are non-dynamical Lagrange multiplier fields, and the Hamiltonian constraint, the momentum one and the constraint associated with the  $U(1)$  gauge transformation are respectively given by

$$\begin{aligned} H_0 &= \frac{1}{\sqrt{\gamma}\phi^2} p_\Phi p_{\Phi^\dagger} - \frac{\sqrt{\gamma}}{2} - \frac{\dot{\phi}^2}{2\sqrt{\gamma}} + \partial_0 \left( \frac{\partial_0(\phi^2)}{2\sqrt{\gamma}} \right) + \frac{\phi^2}{\sqrt{\gamma}} |D_0 \Phi|^2 \\ &\quad - \frac{2\sqrt{\gamma}}{\phi} p_\phi p_\gamma + \frac{2\gamma\sqrt{\gamma}}{\phi^2} p_\gamma^2 + \frac{\sqrt{\gamma}}{2\phi^2} p_A^2, \\ H_1 &= \frac{1}{\gamma} [p_\Phi D_0 \Phi + p_{\Phi^\dagger} (D_0 \Phi)^\dagger] + \frac{1}{\gamma} p_\phi \dot{\phi} - 2\dot{p}_\gamma - \frac{1}{\gamma} p_\gamma \dot{\gamma}, \\ H_2 &= -ie (p_\Phi \Phi - p_{\Phi^\dagger} \Phi^\dagger) - \dot{p}_A. \end{aligned} \quad (16)$$

As is well known, the action can be written as the first-order ADM canonical form by the dual Legendre transformation

$$S = \int dx^1 \left[ \int dx^0 (p_\Phi \Phi' + p_{\Phi^\dagger} \Phi'^\dagger + p_\phi \phi' + p_\gamma \gamma' + p_A A'_0) - H \right]. \quad (17)$$

In order to obtain the correct Hamiltonian which yields the Einstein equations through the Hamilton equations, it is necessary to supplement surface terms to the Hamiltonian (15) [20]. In the formalism at hand, since we take the variation of all the fields to be zero at boundaries, we do not have to add any surface terms to the Hamiltonian.

### 3 Black hole radiation in the interior region

We are now ready to apply the canonical formalism constructed in the previous section for understanding the Hawking radiation [14] from the viewpoint of the internal region of a black hole in quantum gravity. A similar analysis was performed in our previous work [11] where only the region near the apparent horizon was considered from the outset. In the present article, we first work with the whole region bounded between the apparent horizon and the spacetime singularity, make important assumptions on some variables, derive the Wheeler-De Witt equation, and solve it analytically. After performing this procedure, we impose the condition of the vicinity of the apparent horizon for compatibility with field equations. We will see that this method nicely overcomes the problem of the vanishing  $\gamma$  variable near the apparent horizon. In the next section, we will apply the same method to the exterior region bounded between the apparent horizon and the spatial infinity.

To consider the simplest model of the Hawking radiation, let us switch off the  $U(1)$  gauge field and treat with the neutral scalar field by which the gauge constraint  $H_2$  is identically vanishing and the Hamiltonian and momentum constraints reduce to the simpler form

$$\begin{aligned} H_0 &= -\frac{\sqrt{\gamma}}{2} - \frac{\dot{\phi}^2}{2\sqrt{\gamma}} + \partial_0 \left( \frac{\partial_0(\phi^2)}{2\sqrt{\gamma}} \right) + \frac{\phi^2}{\sqrt{\gamma}} (\partial_0 \Phi)^2 - \frac{2\sqrt{\gamma}}{\phi} p_\phi p_\gamma + \frac{2\gamma\sqrt{\gamma}}{\phi^2} p_\gamma^2, \\ H_1 &= \frac{1}{\gamma} p_\Phi \partial_0 \Phi + \frac{1}{\gamma} p_\phi \dot{\phi} - 2\dot{p}_\gamma - \frac{1}{\gamma} p_\gamma \dot{\gamma}. \end{aligned} \quad (18)$$

Moreover, we will use the ingoing Vaidya metric [21] to describe the black hole radiation. The treatment of the case of the outgoing Vaidya metric can be made in a perfectly analogous manner. We therefore define the two-dimensional coordinates  $x^a$  as

$$x^a = (x^0, x^1) = (v - r, r), \quad (19)$$

where  $v$  is the advanced time coordinate. Now let us fix the two-dimensional diffeomorphisms by the gauge conditions

$$g_{ab} = \begin{pmatrix} \gamma & \beta \\ \beta & \frac{\beta^2}{\gamma} - \alpha^2 \end{pmatrix} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & \frac{2M}{r} \\ \frac{2M}{r} & 1 + \frac{2M}{r} \end{pmatrix}. \quad (20)$$

From the gauge conditions (20), the two-dimensional line element takes the form of the Vaidya metric [21]

$$\begin{aligned} ds^2 &= g_{ab} dx^a dx^b, \\ &= -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr. \end{aligned} \quad (21)$$

For a dynamical black hole, we make use of the local definition of the horizon, namely, the apparent horizon, rather than the global one, the event horizon. Note that the apparent horizon is now defined as

$$x^1 = r = 2M(x^0, x^1), \quad (22)$$

where  $M = M(x^0, x^1)$  plays the role of the mass function of a black hole.

Since we treat with a massless scalar field which moves along the null geodesics, it is natural to assume that the scalar field  $\Phi$  depends on only the null coordinate  $v$ , by which the mass function  $M$  also becomes the function of the advanced time coordinate  $v$ . Furthermore, the radial function  $\phi$  is assumed to be a radial coordinate  $r$ . Thus, under the situation at hand, we assume

$$\Phi \approx \Phi(v), \quad M \approx M(v), \quad \phi \approx r. \quad (23)$$

Henceforth, we shall use the symbol  $\approx$  to indicate the equalities holding under the assumptions (23). It will turn out that these assumptions play a critical role in simplifying the diffeomorphism constraints and lead to a solvable model of a quantum black hole.

Here it is valuable to check whether the above assumptions (23) to be compatible with the field equations as follows: The field equations obtained from the action (5) are given by

$$\begin{aligned} -\frac{2}{\phi}\nabla_a\nabla_b\phi + \frac{2}{\phi}g_{ab}\nabla_c\nabla^c\phi + \frac{1}{\phi^2}g_{ab}\partial_c\phi\partial^c\phi - \frac{1}{\phi^2}g_{ab} &= 2\left(\partial_a\Phi\partial_b\Phi - \frac{1}{2}g_{ab}\partial_c\Phi\partial^c\Phi\right), \\ \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b\phi\right) - \frac{1}{2}R\phi &= -\phi\partial_a\Phi\partial^a\Phi, \\ \partial_a\left(\sqrt{-g}\phi^2g^{ab}\partial_b\Phi\right) &= 0. \end{aligned} \quad (24)$$

The Vaidya metric (21) gives us the metric tensor

$$g_{ab} = \begin{pmatrix} g_{vv} & g_{vr} \\ g_{rv} & g_{rr} \end{pmatrix} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 1 \\ 1 & 0 \end{pmatrix}. \quad (25)$$

To check the compatibility of the assumptions (23) with the field equations (24), let us make an ansatz that the variables have the form

$$M \approx M(v), \quad \phi \approx r, \quad (26)$$

but the scalar field is still the function of both  $v$  and  $r$ , i.e.,  $\Phi \approx \Phi(v, r)$ . The first field equation in (24) is satisfied if the following relations hold:

$$\partial_r\Phi \approx 0, \quad \partial_v\Phi \approx \frac{\sqrt{\partial_v M}}{r}. \quad (27)$$

Then, we find the second equation in (24) to be satisfied automatically. Finally, using Eq. (27), the third field equation in (24) reduces to the nontrivial equation

$$2r\partial_v\partial_r\Phi + 2\partial_v\Phi + r\left(1 - \frac{2M}{r}\right)\partial_r^2\Phi \approx 0. \quad (28)$$

This equation (28) as well as the latter relation in (27) require us to limit ourselves to working with the vicinity of the apparent horizon (22) [11]. In other words, for the consistency of the field equations, in addition to the assumptions (23), one has to supplement one more assumption

$$r \approx 2M(v). \quad (29)$$

To put differently, with the assumption (29) we study physics of a black hole near the apparent horizon inside a black hole.

After all, given the assumptions (23) and (29), the field equations become

$$\begin{aligned} \partial_r\Phi &\approx 0, \\ \partial_v\Phi &\approx \frac{\sqrt{\partial_v M}}{r} \approx \frac{\sqrt{\partial_v M}}{2M}, \\ \partial_r\partial_v\Phi &\approx \partial_v\partial_r\Phi \approx -\frac{\sqrt{\partial_v M}}{r^2} \approx -\frac{\sqrt{\partial_v M}}{4M^2}. \end{aligned} \quad (30)$$



Then, the solution is found to be

$$\Phi(v, r) = \left(1 - \frac{2M}{r}\right)^2 \frac{1}{4\sqrt{\partial_v M}} + \int^v dv \frac{\sqrt{\partial_v M}}{2M}. \quad (31)$$

As a result, we have

$$\Phi(v, r) \approx \int^v dv \frac{\sqrt{\partial_v M}}{2M}, \quad (32)$$

which means that one can set  $\Phi(v, r) \approx \Phi(v)$  in the vicinity of the apparent horizon. In this sense, the assumptions (23), if the assumption (29) is added, are at least classically consistent with the field equations (24).

Next we will turn our attention to quantum theory. In quantum gravity, following Dirac [22], one must impose the constraints (18) on the wave functional  $\Psi$  as operator equations to find the physical state. In general, it is very difficult to solve such constraint equations at the same time. In some specific situations, however, the constraints become tractable. For instance, in quantum cosmology, the fundamental equation is the Wheeler-De Witt equation,  $H_0\Psi = 0$ , which is the operator equation associated with the Hamiltonian constraint, since the momentum constraint becomes trivial in cosmology owing to the translation invariance of the universe. Of course, solving the Wheeler-De Witt equation is still a tough work, so some people try to simplify the equation by considering minisuperspace.

In this article, our strategy for solving the operator equations  $H_0\Psi = 0$ ,  $H_1\Psi = 0$  is similar to minisuperspace approach in the sense that we set up the assumptions (23) to simplify the constraints. However, we do not impose the assumption (29) a priori since this assumption is not only unnecessary in order to simplify the constraint equations but also leads to a problem of the dynamical variable  $\gamma$  being zero. Indeed, it is remarkable that the assumptions (23) not only make the momentum constraint agree with the Hamiltonian one up to an overall factor but also reduce the unified operator equation, which is nothing but the Wheeler-De Witt equation, to be a solvable equation in an analytical manner.

With the two-dimensional coordinates (19), the derivative operators take the form

$$\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}\right) \equiv (\partial_0, \partial_1) = (\partial_v, \partial_v + \partial_r). \quad (33)$$

With the help of Eqs. (7) and (20), in case of a real scalar field and vanishing gauge field the assumptions (23) reduce the canonical conjugate momenta (13) to the form

$$\begin{aligned} p_\Phi &\approx -\phi^2 \partial_v \Phi, & p_\phi &\approx -\frac{1}{\frac{2M}{\phi} - 1} \partial_v M + 1 - \frac{M}{\phi}, \\ p_\gamma &\approx -\frac{\phi}{2} \approx -\frac{r}{2}. \end{aligned} \quad (34)$$

Then, the remarkable point is that the momentum constraint becomes identical with the Hamiltonian constraint up to an irrelevant overall factor

$$\sqrt{\gamma} H_0 \approx -\gamma H_1$$

$$\approx \frac{1}{\phi^2} p_\Phi^2 + \left( \frac{2M}{\phi} - 1 \right) \left( p_\phi - 1 + \frac{M}{\phi} \right). \quad (35)$$

This compatibility between the momentum and the Hamiltonian constraints justifies the assumptions (23) in quantum gravity.

The constraint (35) as an operator equation on the wave functional  $\Psi$  gives rise to the Wheeler-De Witt equation

$$\left[ -\frac{1}{\phi^2} \frac{\partial^2}{\partial \Phi^2} + \left( \frac{2M}{\phi} - 1 \right) \left( -i \frac{\partial}{\partial \phi} - 1 + \frac{M}{\phi} \right) \right] \Psi = 0. \quad (36)$$

It is worthwhile to rewrite this Wheeler-De Witt equation as follows:

$$i \frac{\partial \Psi}{\partial T} = \left[ p_\Phi^2 - \frac{2M^2}{e^{2MT} + 1} \tanh(MT) \right] \Psi, \quad (37)$$

where we have defined  $T = -\frac{1}{2M} \log(\frac{2M}{\phi} - 1)$ . This Wheeler-De Witt equation can be interpreted as the Schrodinger equation with the Hamiltonian  $H = p_\Phi^2 - \frac{2M^2}{e^{2MT} + 1} \tanh(MT)$  and the time  $T$  in the superspace at hand. It is of interest to note that the superspace time  $T$  "stops" on the apparent horizon owing to gravitational time dilation. On the other hand, the Hamiltonian has a problematic behavior in that it is not positive semi-definite.

Now it is easy to find a special solution of the Wheeler-De Witt equation (36) by the method of separation of variables. The result is given by

$$\Psi = \left( B e^{\sqrt{A}\Phi} + C e^{-\sqrt{A}\Phi} \right) e^{i \left[ \phi - M \log \phi - \frac{A}{2M} \log \left( \frac{2M}{\phi} - 1 \right) \right]}, \quad (38)$$

where  $A, B$  and  $C$  are integration constants. Provided that the expectation value  $\langle \mathcal{O} \rangle$  of an operator  $\mathcal{O}$  is defined as

$$\langle \mathcal{O} \rangle = \frac{1}{\int d\Phi |\Psi|^2} \int d\Phi \Psi^* \mathcal{O} \Psi, \quad (39)$$

one can calculate the expectation value of mass-loss rate  $\langle \partial_v M \rangle$  by using (34) or (35)

$$\langle \partial_v M \rangle = -\frac{A}{\phi^2}. \quad (40)$$

At this stage, let us substitute the condition (29) for the consistency of field equations. Then, we obtain

$$\langle \partial_v M \rangle = -\frac{k^2}{4M^2}, \quad (41)$$

where we have set  $A = k^2$ . This result precisely coincides with that by Hawking in his semiclassical approach. Accordingly, our result shows that a black hole completely evaporates within a finite time. However, it is worth stressing the difference between the Hawking

approach and the present one: In the Hawking semiclassical approach, the gravitational field is fixed as a classical background, and the matter field is treated only quantum-mechanically. By contrast, our formulation is purely quantum-mechanical even if we have imposed physically plausible assumptions on the scalar field, mass function and the radial field.

Two remarks are in order. First of all, recall that in our previous article [11] we have derived the same result for the expectation value of the mass-loss rate (41) in the interior near the apparent horizon, but we have encountered a difficulty of the dynamical variable  $\gamma$  becoming zero, by which various equalities become singular. Thus we further had to take a regularization such that  $\gamma$  is not strictly zero but takes a small but finite value. It is worth mentioning that in the present formulation this artificial regularization is avoided by imposing the condition (29) only at the final stage.

Second, one should comment on the boundary condition on the wave functional  $\Psi$ . Our physical state, which satisfies the Wheeler-De Witt equation, takes the form

$$\Psi = \left( B e^{|k|\Phi} + C e^{-|k|\Phi} \right) e^{i \left[ \phi - M \log \phi - \frac{k^2}{2M} \log \left( \frac{2M}{\phi} - 1 \right) \right]}. \quad (42)$$

This physical state does not satisfy the Dirichlet boundary condition  $\Psi \rightarrow 0$  for  $|\Phi| \rightarrow \infty$ , nor is its norm  $\int d\Phi |\Psi|^2$  finite. Of course, these requirements might be too strict since we do not have any physical principle to pick up the appropriate boundary conditions, and the present formulation does not provide any information on the correct definition of the inner product. Near the spacetime singularity, because of the huge quantum effects, the matter field  $\Phi$  would fluctuate so strongly that the Dirichlet boundary condition seems to be appropriate to suppress such a unwieldy behavior of the physical state.

## 4 Black hole radiation in the exterior region

Now we will move on to an application of our idea for the Hawking radiation in the exterior region of a black hole in quantum gravity. The same problem has been already attacked by Tomimatsu [13] where the assumption (29) is fully utilized from scratch. In this section, we will use only the assumptions (23) to find the physical state, and impose the condition (29) at the final stage in interpreting the mass-loss rate of a dynamical black hole.

The argument proceeds in a perfectly analogous way to the case of the interior region of a black hole. In the exterior, the ADM splitting of (1+1)-dimensional spacetime is of form [12]

$$g_{ab} = \begin{pmatrix} \frac{\beta^2}{\gamma} - \alpha^2 & \beta \\ \beta & \gamma \end{pmatrix}. \quad (43)$$

The normal unit vector  $n^a$  to the Cauchy hypersurfaces  $x^0 = \text{const}$  reads

$$n^a = \left( \frac{1}{\alpha}, -\frac{\beta}{\alpha\gamma} \right). \quad (44)$$

The trace of the extrinsic curvature is calculated to be

$$K = \frac{\dot{\gamma}}{2\alpha\gamma} - \frac{\beta'}{\alpha\gamma} + \frac{\beta}{2\alpha\gamma^2}\gamma'. \quad (45)$$

In case of a real scalar field and vanishing gauge field, the canonical conjugate momenta  $\pi_\Phi$ ,  $\pi_\phi$ , and  $\pi_\gamma$  are now given by

$$\begin{aligned} \pi_\Phi &\approx \phi^2 \partial_v \Phi, & \pi_\phi &\approx \frac{1}{1 + \frac{2M}{\phi}} \partial_v M + \frac{\frac{2M^2}{\phi^2}}{1 + \frac{2M}{\phi}}, \\ \pi_\gamma &\approx \frac{M}{1 + \frac{2M}{\phi}}. \end{aligned} \quad (46)$$

The momentum constraint turns out to become proportional to the Hamiltonian constraint again

$$\begin{aligned} \sqrt{\gamma}H_0 &\approx \gamma H_1 \\ &\approx \frac{1}{\phi^2} \pi_\Phi^2 - \left(1 + \frac{2M}{\phi}\right) \pi_\phi + \frac{2M^2}{\phi^2}. \end{aligned} \quad (47)$$

This compatibility between the momentum and the Hamiltonian constraints justifies the assumptions (23) in quantum gravity as well.

An imposition of the constraint (47) as an operator equation on the wave functional  $\Psi$  produces the Wheeler-De Witt equation

$$\left[ -\frac{1}{\phi^2} \frac{\partial^2}{\partial \Phi^2} + i \left(1 + \frac{2M}{\phi}\right) \frac{\partial}{\partial \phi} + \frac{2M^2}{\phi^2} \right] \Psi = 0. \quad (48)$$

Then, a special solution of the Wheeler-De Witt equation (48) is given by

$$\Psi = \left( B e^{\sqrt{A}\Phi} + C e^{-\sqrt{A}\Phi} \right) e^{i \frac{A-2M^2}{2M} \log(1 + \frac{2M}{\phi})}, \quad (49)$$

where  $A, B$  and  $C$  are integration constants. As before, the expectation value of mass-loss rate  $\langle \partial_v M \rangle$  is calculated to be

$$\langle \partial_v M \rangle = -\frac{A}{\phi^2}. \quad (50)$$

After the assumption (29) is inserted to this result, we obtain

$$\langle \partial_v M \rangle = -\frac{k^2}{4M^2}, \quad (51)$$

where we have defined  $A = k^2$  again. This result precisely coincides with that obtained in the interior region of a black hole.

Thus we have shown that the result of the mass-loss rate of a black hole owing to the Hawking radiation is equivalent between the interior and the exterior of a black hole. This fact suggests that the Hawking radiation comes from the spacetime singularity as expected. Of course, the physical state satisfying the Wheeler-De Witt equation is different between the interior and the exterior regions of a black hole, but the reason is connected with the fact that the Vaidya metric is not the maximally extended, complete geometry and has a coordinate singularity at the horizon.

As a final comment, let us consider the Birkhoff theorem within the present framework. In classical general relativity, the Birkhoff theorem holds in the spherically symmetric geometry, thereby prohibiting the existence of the graviton. In this article, we have also considered the spherically symmetric geometry, so it is natural to ask ourselves if we could get some information on the Birkhoff theorem in quantum gravity. The reparametrization invariance in two dimensions allows the dynamical variable  $\gamma$  and the radial function  $\phi$  to remain as physical degrees of freedom except for the matter field  $\Phi$ . As seen in the arguments done thus far, the variable  $\phi$  is removed via the assumptions (23) so it does not play the role of a dynamical variable. This reduction of the dynamical degree of freedom could be understood as a result of the momentum constraint. In this context, it is important to notice that the condition (29) is not needed to reduce the momentum constraint. On the other hand, via the gauge conditions (20), the variable  $\gamma$  is related to the mass function  $M(v)$  whose change rate with respect to the advanced time  $v$  is evaluated in this article. In this sense, the gravitational mode is left in the form of the mass function for a dynamical black hole in quantum gravity even if there is no explicit gravitational wave.

## 5 Conclusion

In this article, we have made assumptions (23), thereby the momentum constraint becoming identical with the Hamiltonian one up to an irrelevant overall factor. Moreover, these assumptions made it possible to solve the Wheeler-De Witt equation in an analytical way. As mentioned before, imposing these assumptions on the matter field, mass function and the radial field can be understood naturally from the physical viewpoint: The massless scalar field propagates along the null geodesics, and the mass function of a black hole receives influences from such a scalar field, so they are the functions of only the variable  $v$ ,  $\Phi = \Phi(v)$ ,  $M = M(v)$ . The radial function plays the role of the radius of a black hole at the primitive level, so that it is natural to take  $\phi = r$ . It seems that these assumptions have more profound mathematical meaning rather than mere technical devices. It is known that the advanced time coordinate  $v$ , i.e., the tortoise coordinate, makes the  $r - t$  plane look "conformal" so that conformal field theory can be applied [23]. Then, our assumptions  $\Phi = \Phi(v)$ ,  $M = M(v)$  can be interpreted as the holomorphic (or analytic, or chiral) condition, thereby making the complicated constraints associated with two-dimensional diffeomorphisms be tractable and solvable analytically.

One of motivations in this paper was to avoid an artificial regularization, which was

adopted in our previous paper [11], such that  $\gamma$  takes a small but finite value, and then derive the mass-loss rate of a black hole owing to the Hawking radiation in a more reasonable way. Indeed, without this regularization we have succeeded in deriving the same result as that in [11]. In retrospect, the analysis in this article seems to give the regularization a sound foundation.

There are a lot of works to be done in future. Firstly, it is interesting to generalize the present formulation to the Reissner-Nordstrom black hole where we have to pay attention to the constraint associated with the  $U(1)$  gauge transformation. Secondly, it might be possible to relax the assumption  $\phi = r$  since this assumption is somewhat ad hoc in that  $\phi$  has a possibility of having a more general function of the  $r$  coordinate. Thirdly, in a recent progress on large interior of a black hole [4], the most dominant contribution comes from  $r = \frac{3}{2}M$  hypersurface, which is very close to the horizon  $r = 2M$ . Thus, it would be interesting to construct a model of quantum black holes holding near  $r = \frac{3}{2}M$  rather than the horizon for understanding this issue in a quantum-mechanical way. Finally, it would be valuable to formulate the present formulation in the Kruskal-Szekeres coordinate system since we can consider both the interior and the exterior regions of a black hole at the same time, and might have a smoothly interpolating physical state in the both regions.<sup>4</sup> We wish to return these problems in future.

## Acknowledgements

This work is supported in part by the Grant-in-Aid for Scientific Research (C) No. 25400262 from the Japan Ministry of Education, Culture, Sports, Science and Technology.

## References

- [1] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, JHEP **1302** (2013) 062, arXiv:1207.3123 [hep-th].
- [2] A. Almheiri, D. Marolf, J. Polchinski, J. Sully and D. Stanford, JHEP **1309** (2013) 018, arXiv:1304.6483 [hep-th].
- [3] K. Papadodimas and S. Raju, arXiv:1502.06692 [hep-th].
- [4] M. Christodoulou and C. Rovelli, arXiv:1411.2854 [gr-qc].
- [5] I. Bengtsson and E. Jakobsson, arXiv:1502.01907 [gr-qc].
- [6] H. Kawai and Y. Yokokura, arXiv:1409.5784 [hep-th].

---

<sup>4</sup>The Kruskal-Szekeres coordinate system holds only for eternal black holes, so some extension of which might be necessary in taking account of a dynamical black hole.

- [7] E. Poisson and W. Israel, Phys. Rev. **D 41** (1990) 1796.
- [8] A. Ori, Phys. Rev. Lett. **67** (1991) 789; **68** (1992) 2117.
- [9] P. R. Brady and C. M. Chambers, Phys. Rev. **D 51** (1995) 4177; P. R. Brady and J. D. Smith, Phys. Rev. Lett. **75** (1995) 1256.
- [10] A. Hosoya, Class. Quant. Grav. **12** (1995) 2967.
- [11] A. Hosoya and I. Oda, Prog. Theor. Phys. **97** (1997) 233.
- [12] P. Hajicek, Phys. Rev. **D 30** (1984) 1178.
- [13] A. Tomimatsu, Phys. Lett. **B 289** (1992) 283.
- [14] S. W. Hawking, Comm. Math. Phys. **43** (1975) 199.
- [15] I. Oda, Phys. Lett. **B 409** (1997) 88; Phys. Rev. **D 57** (1998) 2415.
- [16] A. Hosoya and I. Oda, gr-qc/9605069.
- [17] G. D. Birkhoff, "Relativity and Modern Physics", Harvard University Press, Cambridge, 1923.
- [18] C. W. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation", W H Freeman and Co (Sd), 1973.
- [19] R. M. Wald, "General Relativity", The University of Chicago Press, 1984.
- [20] T. Regge and C. Teitelboim, Ann. of Phys. **88** (1974) 296.
- [21] P. C. Vaidya, Proc. Indian Acad. Sci. **A 33** (1951) 264.
- [22] P. A. M. Dirac, "Lectures on Quantum Mechanics", Dover Books on Physics, 2001.
- [23] L. Motl and A. Netzke, Adv. Theor. Math. Phys. **7** (2003) 307, hep-th/030117.